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On the thermodynamic limit of the Mehta–Dyson one-dimensional plasma with long range interaction

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Abstract. Using a careful extrapolation method of the exact results for systems with a small number of particles $N \leq 4$, we estimate the thermodynamic limit of the free energy density and correlation energy for the Mehta–Dyson one-dimensional plasma with long range logarithmic interaction. Our results are then compared with those of the exact solution (the Mehta formula), which is established for every value of the plasma parameter $\gamma = e^2/kT$.

1. Introduction

During the past few years there has been considerable interest in the study of the ν -dimensional Coulomb systems with long range interactions between particles, in particular the ν -dimensional one-component plasma (Hansen and Pollock 1973, Malmberg and O'Neil 1977, Kalman 1978, Prasad and O'Neil 1979, Deutsch *et al* 1982). The one-component plasma consists of an assembly of N identical classical point particles each carrying a charge e and interacting through the ν -dimensional Coulomb potential $\varphi(\mathbf{r})$, the solution of the Poisson equation $\Delta\varphi = -(2\pi)^{\nu/2}\Gamma(\nu/2)^{-1}e^2\delta(\mathbf{r})$, where Δ denotes the ν -dimensional Laplace operator. To ensure charge neutrality, the particles are immersed in a uniform neutralising charge background. It may be said that the one-component plasma is the simplest model of a continuous Coulomb fluid and hence of considerable theoretical importance; in a theoretical context one of the main motivations to investigate the properties of such a system is related to one of the fundamental problems in statistical mechanics of charged particles, namely if classical statistical mechanics may describe a charge ordered state or a crystalline state in a classical model with long range Coulomb forces at sufficiently low temperature, in dimension $\nu > 1$. The two-dimensional case has received much attention lately and a number of exact and approximate theoretical results (Calinon *et al* 1979, Bakshi *et al* 1979, Alastuey and Jancovici 1981) as well as Monte Carlo and molecular dynamics experiments (Choquard *et al* 1980, Caillol *et al* 1982, De Leeuw and Perram 1982) are known. There is some indication for the existence

of a first-order fluid–solid phase transition taking place in the system at low temperature (with the possibility of a weaker form of ordering termed ‘orientational’, i.e. long range correlation of bond angles more than a true crystal ordering, i.e. long range positional correlation) but a theoretical proof is still missing. A model which is similar to the two-dimensional one-component plasma, to be investigated in this work, is the one considered some years ago by Mehta (1967) in connection with his theory of random matrices; this is the Mehta–Dyson one-dimensional model of plasma where the point particles are free to move on the straight line but where they still interact with the two-dimensional genuine logarithmic Coulomb potential; a harmonic potential which attracts each particle independently toward the origin has been added in order to ensure some stability in the system. This one-dimensional model bears some analogy with the two-dimensional one-component plasma: it is also unusual in that a number of exact results are known at some particular values of the temperature; some of the exact results may now be established at any temperature (see for example the Mehta formula for the exact solution of the free energy density (Mehta 1967)); we wish also to mention that the existence of the thermodynamic limit for the free energy cannot be proved by standard methods like the ν -dimensional one-component plasma where the Coulomb force is the genuine ν -dimensional interaction (Lieb and Narnhofer 1975, Sari and Merlini 1976, Albeverio *et al* 1983, Martinelli and Merlini 1984); in relation to the Mehta formula it would also be interesting to investigate the system by Monte Carlo or molecular dynamics techniques in order to test the amount of precision involved. In this work we follow another approach and compute the thermodynamics of the system using and testing a method recently introduced with success in a preliminary treatment of the two-dimensional one-component plasma (Johannesen and Merlini 1983). We first introduce the model and compute the harmonic approximation in § 2; our method of extrapolation involving the exact results of systems with a very small number of particles, i.e. $N \leq 4$, is then used in § 3 where we obtain a very accurate estimate of the thermodynamic limit for the free energy density (the exact solution is derived in the appendix). The same method is also applied independently to compute the correlation energy, which is found in particular to be continuous up to very low temperature; this as expected indicates the absence of any first-order phase transition in the model, a situation very different from that recently found in the two-dimensional one-component plasma. In § 4 we briefly give our concluding remarks.

2. The model and the harmonic approximation

2.1. Definition of the model

The model we consider consists of a gas of N point charges with positions x_1, x_2, \dots, x_N free to move on the infinite *straight line* $-\infty < x < \infty$. The potential energy of the gas is given by (Mehta 1967)

$$H = \frac{1}{2}e^2 \sum_{i=1}^N x_i^2 - e^2 \sum_{i<j}^N \ln|x_i - x_j| \quad (1)$$

where $\{x_i\}$ are the coordinates of the point charges and e is the unit charge. The model is the one-dimensional analogue of the two-dimensional one-component plasma (Sari *et al* 1976). The first term in (1) represents a harmonic potential which attracts each

charge independently towards the point $x = 0$, taken as the origin; the second term represents an electrostatic repulsion between each pair of charges (the two-dimensional long range Coulomb logarithmic potential on the line). The partition function is given by

$$\bar{Q}_N = \int_{\mathbb{R}^{1 \times N}} e^{-\beta H} \{dx_i\}, \quad x_i \in \mathbb{R}^1. \tag{2}$$

In order to obtain an extensive free energy in the thermodynamic limit, $N \rightarrow \infty$, we define $\forall N$ the reduced partition function

$$Q_N = \int_{\mathbb{R}^{1 \times N}} \{dx_i\} \exp[-\beta(H - H_0)] / (N!)^{1/2} \tag{3}$$

where $\gamma = (kT)^{-1} e^2 = \beta e^2$ is the coupling parameter and H_0 is the ground state energy. H_0 is determined by the unique equilibrium configuration $\{x_{i0}\}$ given by

$$\frac{\partial H}{\partial x_{i0}} = x_{i0} - \sum_{j \neq i} \frac{(x_{i0} - x_{j0})}{|x_{i0} - x_{j0}|^2} = 0 \quad \forall i = 1, 2, \dots, N. \tag{4}$$

$\{x_{i0}\}$ are the zeros of the Hermite polynomial of order N and H_0 is given $\forall N$ by (Mehta 1967)

$$H_0 = \frac{1}{4} N(N - 1)(1 + \ln 2) - \frac{1}{2} \sum_{i=1}^N i \ln i. \tag{5}$$

Notice that $H(\{x_i\}) \geq H_0, \forall \{x_i\} \neq \{x_{i0}\}$.

2.2. The strong γ limit and the harmonic approximation

Contrary to the two-dimensional Coulomb plasma, where the logarithmic interaction is the genuine two-dimensional Coulomb interaction, it is not possible here to derive rigorous lower and upper bounds for the free energy, using the stability property given by (5) or by standard use of the Jensen inequality. The first aim is then to compute exactly the harmonic approximation, which should be very accurate at low temperature (high value of γ). To do this, we consider (3) and expand the Hamiltonian around the equilibrium positions $\{x_{i0}\}$, the solution of (4), and retain only quadratic terms in the displacements $\{\xi_i\}$, i.e. with $x_i = x_{i0} + \xi_i, \forall i$, we have

$$\begin{aligned} Q_N &= \int \{dx_i\} \exp[-\gamma(H - H_0)] / (N!)^{1/2} \\ &= N! \frac{\int \{d\xi_i\}}{(N!)^{1/2}} \exp \left\{ -\gamma \left[\sum_i \frac{1}{2} \xi_i^2 + \sum_i x_{i0} \xi_i - \frac{1}{2} \sum_{i < j} \ln \left(1 + \frac{\xi_{ij}^2 + 2x_{ij0} \xi_{ij}}{|x_{ij0}|^2} \right) \right] \right\} \end{aligned} \tag{6}$$

where $x_{ij0} = x_{i0} - x_{j0}$ and $\xi_{ij} = \xi_i - \xi_j$. Notice that the factor $N!$ in (6) comes from the fact that there are $N!$ possible ways to place the N charges at the equilibrium positions $\{x_{i0}\}$. Expanding the \ln in (6), retaining only quadratic terms in the displacements $\{\xi_i\}$, and introducing the change of variable $\xi_i(\gamma)^{1/2} = \alpha_i$, and writing $\alpha_{ij} = \alpha_i - \alpha_j$, we obtain

the harmonic approximation which reads

$$\begin{aligned}
 Q_N^h &= \frac{(N!)^{1/2}}{(\gamma)^{N/2}} \int \{d\alpha_i\} \exp \left[-\left(\frac{1}{2} \sum_i \alpha_i^2 + \frac{1}{2} \sum_{i < j} \frac{(\alpha_{ij})^2}{|x_{j0}|^2} \right) \right] \\
 &= [(N!)^{1/2}/(\gamma)^{N/2}] \int \{d\alpha_i\} \exp[-\frac{1}{2}(\alpha A \alpha)] \\
 &= (N!)^{1/2} (2\pi)^{N/2} / (\gamma)^{N/2} (\text{Det}|A^{(N)}|)^{1/2}
 \end{aligned} \tag{7}$$

where $A^{(N)}$ is the $N \times N$ square matrix associated with $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_N)$.

Det $A^{(N)}$ should be of the order $N!$; an indication for this (a mathematical proof will probably require the explicit use of some properties of the Hermite polynomials) is the following: if $\{\lambda_i^{(N)}\}$, $i = 1, 2, 3, \dots, N$, are the eigenvalues of $A^{(N)}$, then from (7), we have that

$$\text{Tr } A^{(N)} = \sum_{i=1}^N \lambda_i^{(N)} = N + \sum_{i \neq j} |x_{i0} - x_{j0}|^{-2}. \tag{8}$$

Since the equilibrium positions $\{x_{i0}\}$ satisfy the equation (Mehta 1967)

$$\sum_{i=1}^N x_{i0}^2 = \sum_{i \neq j} |x_{j0}|^{-2}$$

we obtain

$$\sum_{i=1}^N \lambda_i^{(N)} = N + \frac{1}{2} N(N-1) = \frac{1}{2} N(N+1) = \sum_{i=1}^N i \quad \forall N$$

so that $\lambda_i^{(N)} = i$, $i = 1, 2, 3, \dots, N$, should be the eigenvalues of $A^{(N)}$, $\forall N$ and $\text{det } A^{(N)} = N!$. In fact, an explicit computation shows that the $\{\lambda_i^{(N)}\}$ are as above for N up to 4. In any case, for every N (see appendix), we obtain the harmonic approximation, which reads

$$Q_N^h = \frac{(N!)^{1/2} (2\pi)^{N/2}}{(\gamma)^{N/2} (N!)^{1/2}} = (2\pi)^{N/2} \frac{1}{(\gamma)^{N/2}} \quad \forall N \text{ and } \gamma \text{ very large.} \tag{9}$$

The extensive harmonic approximation to the free energy defined by $Q_N^h = \exp(-\beta f_h N)$ reads

$$\beta f_h / \gamma = (\ln \gamma - \ln 2\pi) / 2\gamma \tag{10}$$

and is similar to that computed for the two-dimensional one-component plasma.

3. Extrapolation method

To obtain our estimate for the free energy density and the correlation energy of the system in the thermodynamic limit, we use a very simple method of extrapolation introduced recently for the two-dimensional one-component plasma (Johannesen and Merlini 1983). Given an observable O_N (in this paper the free energy density or the correlation energy of the model with N particles), the method consists in computing *exactly* $O_N(\gamma)$, for identical systems just consisting of very few particles, i.e. $N = 1, 2, 3, 4$, $N \leq 4$, or $N \leq 5$, for a large range of the coupling constant γ .

The amount of information contained in $O_N(\gamma)$, $N \leq 4$, and a careful analysis of the way in which the asymptotic limit $\lim_{N \rightarrow \infty} O_N = O$ is approached (which, for the two-dimensional one-component plasma and the present model, may be derived rigorously at the particular values $\gamma = 2$) suggest the ansatz

$$O_N = 0 + ax + bx \ln x + cx^2 + \dots \tag{11}$$

where $x = 1/N$, and a, b, c, \dots depend on γ . a, b, c are determined by the exact knowledge of O_N , $N \leq 4$. We now apply the method to obtain our estimate for the free energy density in the thermodynamic limit.

3.1. The free energy density

We consider identical systems consisting of $N \leq 4$ particles. What we need is the exact partition function Q_N , for every γ (equation (3)). For the two-dimensional model this may be done by computer (Johannesen and Merlini 1984); in the one-dimensional case here, some exact analytical results are available. In fact for $N \leq 3$ and $\forall \gamma$ it has been proved that (Mehta 1967)

$$Q_N = \left(\frac{2\pi}{\gamma}\right)^{N/2} \frac{1}{(\gamma)^{N(N-1)\gamma/4}} \exp(\gamma H_0) \prod_{i=1}^N \frac{(\frac{1}{2}\gamma i!)}{[(\frac{1}{2}\gamma)!]^N} \frac{1}{(N!)^{1/2}} = \exp(-\beta f_N N). \tag{12}$$

For $N = 4$ we have also verified (12) up to $\gamma = 8$. In the following we also assume that (12) is correct for $N = 4$ and every γ . (This may also be verified using computer calculations (see also the acknowledgments and appendix).) Moreover, it is known that in the one-dimensional case (12) has been proved to be correct for all N at $\gamma = 1, 2, 4$ (Mehta 1967). With βf_N resp. βf^h , as defined by (12) and (10), the excess free energy $\beta \Delta f_N$ for $N \leq 4$ is given by

$$\beta \Delta f_N / \gamma = (\beta f_N - \beta f^h) / \gamma = \frac{1}{2} \ln N! / \gamma^N + \frac{1}{4}(N-1)(\ln \gamma - 1 - \ln 2) + \frac{1}{2N} \sum_{i=1}^N i \ln i + \frac{1}{\gamma^N} \left\{ N \ln \left[\left(\frac{\gamma}{2}\right)! \right] - \ln \prod_{i=1}^N \left(\frac{\gamma}{2} i!\right) \right\}. \tag{13}$$

The computations of $\gamma^{-1} \Delta f_N(\gamma)$ for $N = 1, 2, 3, 4$ at some particular values of γ are given in table 1.

Table 1.

γ	$\gamma^{-1} \Delta f_1$	$\gamma^{-1} \Delta f_2$	$\gamma^{-1} \Delta f_3$	$\gamma^{-1} \Delta f_4$
2	0	$0.993\ 019 \times 10^{-2}$	$1.551\ 373 \times 10^{-2}$	$1.911\ 718 \times 10^{-2}$
4	0	$0.256\ 875 \times 10^{-2}$	$0.400\ 121 \times 10^{-2}$	$0.493\ 400 \times 10^{-2}$
20	0	$1.041\ 06 \times 10^{-4}$	$1.619\ 522 \times 10^{-4}$	$1.995\ 550 \times 10^{-4}$
40	0	$2.603\ 787 \times 10^{-5}$	$4.050\ 395 \times 10^{-5}$	$4.990\ 707 \times 10^{-5}$
60	0	$1.157\ 3324 \times 10^{-5}$	$1.800\ 030 \times 10^{-5}$	$2.218\ 243 \times 10^{-5}$
80	0	$0.651\ 0179 \times 10^{-5}$	$1.012\ 698 \times 10^{-5}$	$1.247\ 792 \times 10^{-5}$
100	0	$0.416\ 657 \times 10^{-5}$	$0.648\ 135 \times 10^{-5}$	$0.798\ 595 \times 10^{-5}$

Using the above values and our ansatz (11) with $c = 0$, we have than obtained our estimate for the free energy density in the thermodynamic limit as given in table 2 ($\gamma^{-1}\Delta f(1, 2, 3)$ and $\gamma^{-1}\Delta f(2, 3, 4)$ obtained with $N = 1, 2, 3$ resp. $N = 2, 3, 4$). In table 3 we given our estimate $\gamma^{-1}\Delta f(1, 2, 3, 4)$ obtained with $N = 1, 2, 3, 4$ and the thermodynamic limit $\gamma^{-1}\Delta f_M$, as it follows from Mehta's formula, to be discussed below. As expected and as may be seen from tables 2 and 3, the estimate of table 3 obtained with $N = 1, 2, 3, 4$ is the best one.

Table 2.

γ	$\gamma^{-1}\Delta f(1, 2, 3)$	$\gamma^{-1}\Delta f(2, 3, 4)$
2	0.036 320 06	0.038 546 3682
4	0.009 302 517	0.009 848 067 95
20	0.000 375 503 67	0.000 397 164 56
40	0.000 939 0413	0.000 099 318 04
60	0.000 041 7375	0.000 044 143 57
80	0.000 023 4778	0.000 024 831 20
100	0.000 015 025 93	0.000 015 892 10

Table 3.

γ	$\gamma^{-1}\Delta f(1, 2, 3, 4)$	$\gamma^{-1}\Delta f_M$
2	0.040 009 273	0.040 530 733
4	0.010 206 550	0.010 335 173
20	0.000 411 397	0.000 416 5281
40	0.000 102 8755	0.000 104 157
60	0.000 045 724 61	0.000 046 2945
80	0.000 025 7205	0.000 026 041
100	0.000 016 461 26	0.000 016 666

3.2. Mehta's conjecture

From a conjecture of Mehta (1967), the thermodynamic limit for the free energy density is given by

$$\lim_{N \rightarrow \infty} Q_N = (2\pi/\gamma)^{N/2} \exp\{N[(\frac{1}{2}\gamma + \frac{1}{2}) \ln(\frac{1}{2}\gamma) - \frac{1}{2}\gamma + \frac{1}{2} \ln 2\pi - \ln(\frac{1}{2}\gamma!)]\} \tag{14}$$

so that

$$\begin{aligned} \Delta f_M/\gamma &= (\beta f_\infty - \beta f^h)/\gamma \\ &= \gamma^{-1}[\ln(\frac{1}{2}\gamma!) - (\frac{1}{2}\gamma + \frac{1}{2}) \ln(\frac{1}{2}\gamma) - \frac{1}{2}\gamma + \frac{1}{2} \ln(2\pi)]. \end{aligned} \tag{15}$$

Equation (14), for every γ , follows from the result given in the appendix. The strong coupling limit $\gamma \rightarrow \infty$ in (15) coincides exactly with our harmonic approximation given by (10), i.e. $\Delta f/\gamma = 0$. The anharmonic corrections in the strong coupling limit from (15) are obtained using the Stirling formula applied to $(\gamma/2)!$, which gives

$$\frac{\Delta f_M}{\gamma}(\gamma \rightarrow \infty) = 0 + \frac{1}{6\gamma^2} - \frac{1}{45\gamma^4} + \dots \tag{16}$$

This shows that the harmonic approximation is ‘almost exact’ for $\gamma \geq 100$. As an example, at $\gamma = 100$ the absolute error for $\beta f / \gamma$ given by this approximation is 0.166×10^{-4} while the relative error is 0.1%. On the other hand, at the same value $\gamma = 100$ our estimate underestimates the free energy $\beta f / \gamma$ only by the absolute amount 2×10^{-7} (see table 3), with a relative error of only $10^{-3}\%$. Our estimate for the free energy turns out to be better than the harmonic approximation in the whole range of γ . As is seen from table 3, our method of extrapolation with $N = 1, 2, 3, 4$ particles gives an excellent estimate of the free energy in the thermodynamic limit. In the range $\gamma > 60$ the absolute error is $< 10^{-6}$ and the relative error to the free energy $\beta f / \gamma$ is smaller than 10^{-5} . Moreover it may be checked that the errors decrease as γ increases. Our estimate for f agrees well with the exact solution at any temperature derived in the appendix.

Our results explicitly show how a very accurate description of the thermodynamic limit of the free energy density may be obtained using the amount of information from systems with a very small number of particles, in conjunction with a careful ansatz. The plot of our estimate for f is given in figure 1.

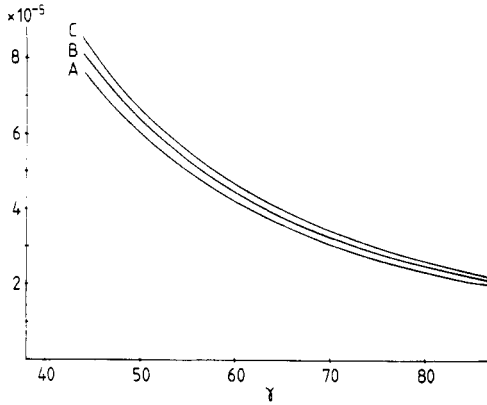


Figure 1. The curve $\gamma^{-1}\Delta f = \gamma^{-1}f_1^{ex}$ (A) (extrapolation with $N = 2, 3, 4$) from table 2, the curve $\gamma^{-1}\Delta f = \gamma^{-1}f_2^{ex}$ (B) (extrapolation with $N = 1, 2, 3, 4$) from table 3 and $\gamma^{-1}\Delta f_M = \gamma^{-1}f$, (C) equation (15) (Mehta’s formula), as a function of γ .

3.3. The correlation energy

The same method of extrapolation (11) may be used to obtain an estimate for the correlation energy in the thermodynamic limit. Using (12) for $N \leq 4$, and defining

$$E_N = -N^{-1}(\partial/\partial\gamma) \ln Q_N, \tag{17}$$

it may be easily seen from (12) that for $N \leq 4$,

$$E_N = \frac{N-1}{N} E_{N-1} + \frac{1}{4Nz} + \frac{N-1}{2N} \ln z + \frac{1}{2} \ln N + \frac{1}{2N} [\psi(z+1) - N\psi(zN+1)] \tag{18}$$

where $E_1 = 1/4z$, $z = \gamma/2$ and $\psi(z) = \Gamma'(z)/\Gamma(z)$. (Notice $\Gamma(n) = (n-1)!$.) On the other hand, the thermodynamic limit E_M , for the correlation energy, using the exact result given by (15), reads

$$E_M = \frac{1}{2} [\psi(z+1) - \ln z]. \tag{19}$$

Using (18), (11) with $N = 1, 2, 3, 4$, we computed our estimate for the correlation energy $E(1, 2, 3, 4)$ which is given in table 4, together with the values of E_M computed with (19).

Table 4.

γ	$E(1, 2, 3, 4)$	E_M
2	0.211 935 889	0.211 392 167
4	0.114 948 085	0.114 818 577
20	0.024 588 879	0.024 583 74
40	0.012 397 141	0.012 395 859
60	0.008 287 612	0.008 287 042
80	0.006 224 280	0.006 223 960
100	0.004 983 5392	0.004 983 334

As for the free energy density, the results of table 4 show an excellent agreement between our estimate and the thermodynamic limit as given by Mehta's formula. Here too, it may easily be verified that at $\gamma = 100$ the absolute absolute error for E is smaller than 10^{-6} while the relative error is 4×10^{-5} . This may be compared with the absolute resp. relative error given by the harmonic approximation (10) at $\gamma = 100$ given by 1.67×10^{-5} resp. 3.3×10^{-4} , which is less accurate. For illustration the results of table 4 are plotted in figure 2.

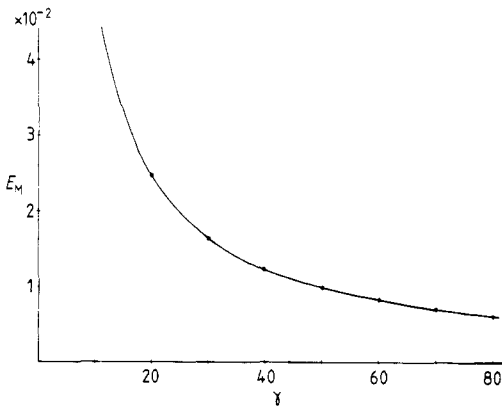


Figure 2. The correlation energy E_M given by (19) (Mehta's formula) and the estimate $E(1, 2, 3, 4)$ from table 4 (full circles).

4. Conclusions

In this work we have used a method of extrapolation to compute thermodynamics of the Mehta–Dyson one-dimensional model of plasma with long range interaction. The amount of statistical information contained in the exact results of the phase space of small systems consisting of just a few particles $N \leq 4$ allows a very accurate estimate of the thermodynamic properties of the model in a wide range of temperature, and the results constitute a test for the usefulness of the method we have followed. The

harmonic approximation is almost exact in the temperature range $\gamma > 100$; nevertheless our estimates with $N = 1, 2, 3, 4$ turn out to give a better description of the thermodynamics in the whole range of temperature considered. Although the thermodynamic functions are analytic in the infinite volume limit, the model is well suited to test the various approximations usually made for Coulomb fluids; and the method may be employed further to investigate the amount of precision involved in ideal Monte Carlo or molecular dynamic computations. Moreover the method is expected to yield accurate computations and estimates in the more difficult two-dimensional case where a first-order phase transition is expected to take place in the system at low temperature.

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Appendix

A1. Proof of Mehta’s formula (exact solution) for all values of the coupling constant γ ((12 and (14))

The formula of Mehta is given by

$$\begin{aligned} \psi_N(\gamma) &= \int_{\mathbb{R}_1 \otimes N} dx_1 \dots dx_N e^{-\gamma H} \\ &= (2\pi)^{N/2} \gamma^{-N/2} \gamma^{-\gamma N(N-1)/4} \prod_{k=1}^N \left(\frac{\Gamma(1 + \frac{1}{2}\gamma k)}{\Gamma(1 + \frac{1}{2}\gamma)} \right) \end{aligned} \tag{A1}$$

where $\Gamma(1+x) = x!$ and $H = \frac{1}{2} \sum_{i=1}^N x_i^2 - \sum_{i < j=1}^N \ln|x_i - x_j|$.

(A1) was proven for $\gamma = 1, 2, 4$ and for every N by Mehta. One can prove (A1) for every γ , using a result on multiple integrals given by Selberg (1944), which reads

$$\begin{aligned} \phi_N(\gamma) &= \int_0^1 \dots \int_0^1 dx_1 dx_2 \dots dx_N \prod_{i=1}^N x_i^{\alpha-1} \prod_{i=1}^N (1-x_i)^{\beta-1} \prod_{i < j=1}^N |x_i - x_j|^{2z} \\ &= \prod_{k=1}^N \left(\frac{\Gamma(1+kz)}{\Gamma(1+z)} \frac{\Gamma[\alpha + (k-1)z] \Gamma[\beta + (k-1)z]}{\Gamma(\alpha + \beta + (k+N-z)z)} \right). \end{aligned} \tag{A2}$$

We then consider (A2) with $z = \gamma/2$, $\alpha = \beta = \frac{1}{2}\gamma L^2 + 1$ and the change of variable $x^i = (L + y_i)/L$ so that $y_i \in (-L, +L)$. With (A2), we then obtain

$$\begin{aligned} \phi_N(N, z = \frac{1}{2}\gamma, \alpha = \beta = \frac{1}{2}\gamma L^2 + 1) \\ = \int_{-L}^{+L} \dots \int_{-L}^{+L} dy_1 dy_2 \dots dy_N \prod_{i=1}^N \left(1 - \frac{y_i^2}{L^2} \right)^{\gamma L^2 N/2} \prod_{i < j=1}^N |y_i - y_j| \end{aligned}$$

$$\begin{aligned} &\times 2^{-N\gamma L^2} (2L)^{-N} (2L)^{-\gamma N(N-1)/2} \\ &= \left[\prod_{k=1}^N \left(\frac{\Gamma(1 + \gamma z/2)}{\Gamma(1 + \gamma/2)} \right) \right] \left[\prod_{k=1}^N \left(\frac{\Gamma^2[\frac{1}{2}\gamma L^2 + 1 + (k-1)\mu/2]}{\Gamma[\gamma L^2 + 2 + (k+N-2)\gamma/2]} \right) \right] \end{aligned}$$

so that

$$\begin{aligned} &\int_{-L}^{+L} \dots \int_{-L}^{+L} dy_1 dy_2 \dots dy_N \left[\prod_{i=1}^N \left(1 - \frac{y_i^2}{L^2} \right)^{\gamma L^2/2} \right] \left(\prod_{i < j=1}^N |y_i - y_j|^\gamma \right) \\ &= \prod_{k=1}^N \frac{\Gamma(1 + \frac{1}{2}\gamma k)}{\Gamma(1 + \frac{1}{2}\gamma)} 2^{N\gamma L^2} (2L)^N (2L)^{\gamma N(N-1)/2} \\ &\quad \times \prod_{k=1}^N \left(\frac{\Gamma^2[\frac{1}{2}\gamma L^2 + 1 + (K-1)\gamma/2]}{\Gamma[\gamma L^2 + 2 + (k+N-2)\gamma/2]} \right). \end{aligned} \tag{A3}$$

Using Stirling’s formula $x! \cong \exp[(x + \frac{1}{2}) \ln x - x + \frac{1}{2} \ln 2\pi + o(1/x)]$ in (A3) we have that

$$\begin{aligned} &\int_{-L}^{+L} \dots \int_{-L}^{+L} \prod_{i=1}^N dy_i \prod_{i=1}^N \left(1 - \frac{y_i^2}{L^2} \right)^{\gamma L^2/2} \prod_{i < j=1}^N |y_i - y_j|^\gamma \\ &= \left(\prod_{k=1}^N \frac{\Gamma(1 + \gamma k/2)}{\Gamma(1 + \gamma/2)} \right) \frac{(2\pi)^{N/2} e^{o(L)}}{\gamma^{N/2} \gamma^{\gamma N(N-1)/4}} \end{aligned}$$

As $L \rightarrow \infty$, the first term converges to $\psi_N(\gamma)$ and $o(L)$, on the right-hand side, converges to 0. We thus obtain

$$\psi_N(\gamma) = \left(\prod_{k=1}^N \frac{\Gamma(1 + \frac{1}{2}\gamma k)}{\Gamma(1 + \frac{1}{2}\gamma)} \right) \frac{(2\pi)^{N/2}}{\gamma^{N/2} \gamma^{\gamma N(N-1)/4}} \tag{A4}$$

which is (A1), and Mehta’s formula is proved for every γ . So, the Mehta–Dyson model of plasma with long range interaction is another example of an exactly solvable model of statistical mechanics. It also follows from (A1) that the free energy is analytic in γ , so that no phase transition occurs.

A2. The harmonic approximation

We now derive the harmonic approximation (9) using (A1) above. Following (6) and (7), the harmonic approximation is given by

$$\begin{aligned} Q_N^h(\gamma) &= \lim_{\gamma \rightarrow \infty} Q_N(\gamma) \\ &= \lim_{\gamma \rightarrow \infty} \int \dots \int dx_1 \dots dx_N \exp[-\gamma(H - H_0)] / \sqrt{N!}. \end{aligned}$$

From (A1), we have that

$$\begin{aligned} \lim_{\gamma \rightarrow \infty} \psi_N(\gamma) &= \lim_{\gamma \rightarrow \infty} \int_{-\infty}^{+\infty} \dots \int e^{-\gamma H} \{dx_i\} \\ &= \lim_{\gamma \rightarrow \infty} \prod_{k=1}^N \frac{\Gamma(1 + \frac{1}{2}\gamma k)}{\Gamma(1 + \frac{1}{2}\gamma)} \frac{(2\pi)^{N/2}}{\gamma^{N/2} \gamma^{\gamma N(N-1)/4}}. \end{aligned} \tag{A5}$$

Using Stirling's formula $x! \cong \exp[(x + \frac{1}{2}) \ln x - x + \frac{1}{2} \ln 2\pi]$ in (A5), we obtain

$$\frac{\psi_N(\gamma)}{\sqrt{N!}} \cong \frac{\exp[\sum_{k=1}^N (\frac{1}{2}\gamma k + \frac{1}{2}) \ln(\frac{1}{2}\gamma k) - \sum_{k=1}^N \frac{1}{2}\gamma k + \frac{1}{2}N \ln 2\pi + o(1/\gamma)]}{\exp[N(\frac{1}{2}\gamma + \frac{1}{2}) \ln \frac{1}{2}\gamma - \frac{1}{2}N\gamma + \frac{1}{2}N \ln 2\pi]} (N!)^{1/2} (\gamma)^{\gamma N(N-1)/4} \left(\frac{2\pi}{\gamma}\right)^{N/2}$$

where $o(1/\gamma) \rightarrow 0$ as $\gamma \rightarrow \infty$.

Explicitly

$$\psi_N(\gamma)/(N!)^{1/2} \cong (2\pi/\gamma)^{N/2} \exp(-\gamma H_0), \quad \text{as } \gamma \rightarrow \infty,$$

where H_0 is given by (5). We finally obtain

$$Q_N^h(\gamma) = \lim_{\gamma \rightarrow \infty} Q_N(\gamma) = (2\pi/\gamma)^{N/2}$$

which proves (9) at very low temperatures (large γ). (This of course does not prove that $\lambda_i^{(N)} = i$, $i = 1, 2, \dots, N$, $\forall N$, but in particular proves that $\text{Det}|A^{(N)}| = N!$, $\forall N$, i.e. the harmonic approximation as we had conjectured on the basis of the exact results for $N = 1, 2, 3, 4$.)

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